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On the sensitivity matrix of the Nash bargaining solution

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Abstract In this note we provide a characterization of a subclass of bargaining problems for which the Nash solution has the property of disagreement point monotonicity. While the original d-monotonicity axiom and its stronger notion, strong d-monotonicity, were introduced and discussed by Thomson (J Econ Theory, 42: 50–58, 1987), this paper introduces local strong d-monotonicity and derives a necessary and sufficient condition for the Nash solution to be locally strongly d-monotonic. This characterization is given by using the sensitivity matrix of the Nash bargaining solution w.r.t. the disagreement point d. Moverover, we present a sufficient condition for the Nash solution to be strong d-monotonic.

Keywords Nash bargaining solution · d-monotonicity · Diagonally dominant Stieltjes matrix

1 Introduction

In this note we introduce the notion of local strong d-monotonicity for solutions of bargaining problems. Thomson introduced and discussed in (Thomson 1987) the disagreement point monotononicity property (*d-monotonicity*) for solutions of bargaining problems. This property states that, if some agent increases his disagreement point (also called threat-point) while the threat-point of the other players remains

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constant then this agent's payoff increases (or at least not decreases). He also considered *strong d-monotonicity*, which states that not only this agent's payoff does not decrease but also the payoffs of the other agents should not increase. Thomson shows by means of a counterexample that the Nash solution does not satisfy this notion of strong d-monotonicity.

This notion of d-monotonicity is a global property in the sense that this property should hold for every positive increment of the threat-point at every threat-point d.

We will consider here the local version of this property. That is, we are interested in the effect of changes on the point of agreement for a fixed feasible set if one (arbitrarily chosen) player unilaterally changes his disagreement point by a small (positive) amount. If this player is the only one who gains and this property holds irrespective of which player alters his threat-point, d, we call the bargaining solution *locally strongly d-monotonic* at d.

Given some threat-point and the corresponding bargaining point, this notion tells us something about the stability of the realized bargaining point. This, in the following sense. Assume that the threat-point can be controlled to some extent by an exogenous authority (e.g., a European commission who might consider to change some directives which might favor some outside options of participating countries). If the bargaining point is locally strongly d-monotonic at d, then whenever this threat-point is changed at one entry only, this action will be disapproved by all other players. This, in contrast to the case that such a change in the threat-point is favourable for some other player(s) too. In that case, it is rational for that (those) other player(s), at least, to not be against such a change. So, fewer players will be against a reopening of bargaining in such a case. In this sense, the threshold to reopen the process will be lower, and the bargaining point might be called less stable.

So, this notion of local strong d-monotonicity can be viewed as a new independent axiom for a bargaining solution which implies stability. We give in Sect. 3 below a necessary and sufficient condition of domain restriction over which the Nash solution has the property of local strong d-monotonicity. Furthermore, we present in this section a sufficient condition for strong d-monotonicity.

Section 2 introduces some notation and preliminary results, whereas Sect. 4 considers some examples. Finally Sect. 5 concludes.

2 Preliminaries

Following Thomson and Lensberg (1989), an *n-person bargaining problem* is a pair (S, d), where $S \subset \mathbb{R}^n$ is called the *feasible set*, $\mathbb{R}^n$ the utility space and d the *disagreement point*.

Thomson considers two classes of bargaining problems: (1) $\overline{\Sigma}^n$, where the feasible set S is convex, compact and such that there exists a $x \in S$ with $x > d$ (here we use the vector inequality notation); and (2) Σ^n, which is a subclass of $\overline{\Sigma}^n$, the so-called class of comprehensive bargaining problems. This subclass is obtained by considering just those elements in S satisfying the additional property that whenever $x \in S$ and $d < \overline{x} \leq x$, then $\overline{x} \in S$.

We will consider in this paper a subclass Σ_P^n of Σ^n. We assume that the (fixed) feasible set satisfies the additional requirement that the set P of (weak) Pareto optimal solutions can be described by a smooth strictly concave function φ, that is $\Sigma_P^n = \{(x_1, \ldots, x_n)^T \in \Sigma^n \mid x_i \geq d_i, x_n \leq \varphi(x_1, \ldots, x_{n-1})$, and whenever $x \in \Sigma_P^n$ and $d \leq y \leq x$, then $y \in \Sigma_P^n\}$. Here v^T denotes the transpose of a vector/matrix v. This class of problems (for larger classes of bargaining problems, see e.g., Peters 1992 or Thomson and Lensberg 1989) is particularly relevant to applied economics (see e.g., the literature on policy coordination Petit 1990; de Zeeuw and van der Ploeg 1991; Ghosh and Masson 1994; Douven and Engwerda 1995; Plasmans and Engwerda 2006).

Given this class of n-person problems, a *solution* is a function F associating with every (S, d) in this class the point of agreement $F(S, d) \in S$. Since we consider here a fixed feasible set, the dependence of F on S will be omitted. F is called the Nash solution, N, if for every fixed pair (S, d), $F(S, d)$ is assigned the point where the product $\Pi(x_i - d_i)$ is maximized for $x \in S$ with $x \geq d$.

For notational convenience **n** denotes the set $\{1, \ldots, n\}$. Furthermore, I is the identity matrix, e_i the ith standard basis vector in $\mathbb{R}^n$, e the vector $(1, \ldots, 1)^T$ and $\underline{0}$ the zero vector $(0, \ldots, 0)^T$. The dimension of these vectors will be clear from the context. Furthermore, diag(a_i) denotes a diagonal matrix with as its ith diagonal entry a_i; $(A|B)$ the extended matrix of A and B; and sgn(a) the sign of the number a. If $x := (x_1, \ldots, x_n)$ is a vector, x_- is the truncated vector $(x_1, \ldots, x_{n-1})$. $\varphi_i^{'}$ denotes the ith partial derivative of φ.

The property of local strong d-monotonicity with respect to the disagreement point d is now formalized as follows:

Definition 1 A bargaining solution F on Σ_P^n is called *locally strongly d-monotonic* at a problem $(S, d) \in \Sigma_P^n$, if F is differentiable in d, and for all i and $j \neq i$, $\frac{\partial F_j(S,d)}{\partial d_i} \leq 0$ and $\frac{\partial F_i(S,d)}{\partial d_i} \geq 0$.

In the ensueing analysis the set of so-called *M-matrices* arise in a natural way. An M-matrix is an $n \times n$ matrix with nonpositive off-diagonal entries whose inverse exists and is entry-wise nonnegative. Symmetric M-matrices are called *Stieltjes matrices*. From Berman and Plemmons (1994, pp. 141), we recall the following result.

Lemma 1 (1) *Symmetric M-matrices are positive definite.*
(2) *Symmetric positive definite matrices with nonpositive off-diagonal entries are M-matrices.*

Unfortunately, the inverse of a nonsingular nonnegative matrix is not in general an M-matrix. In literature the problem has been addressed to characterize all matrices that have this property. This turns out to be a difficult problem. A class of matrices that satisfy this property are e.g., the so-called strictly ultrametric matrices (see Nabben and Varga 1994, Nabben 2000).

Finally, we call a symmetric square matrix $A = (a_{ij})$ diagonally dominant if $|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|$, for all i.

3 Theoretical results

By assumption, the Nash bargaining solution $x^N := (x_1^N, \ldots, x_n^N)$ is the argument that solves the maximization problem

$$\max_{x_-} f(x_-) := \max_{x_-} \Pi_{i\in\mathbf{n-1}}(x_i - d_i)(\varphi(x_-) - d_n),$$

where $\varphi'_i < 0$ and φ'' is negative definite.

This maximization problem has, according to Nash (1950), exactly one solution. Obviously, this solution lies not on the edge of the Pareto frontier P of S, i.e., it is an interior point of P. Thus, the first order conditions yield that the Nash bargaining solution is uniquely determined by:

$$g_i(x_-^N, d) = 0, \quad \forall i \in \mathbf{n-1}, \tag{1}$$

where $g_i(x_-, d) := \varphi(x_-) - d_n + (x_i - d_i)\varphi'_i(x_-),\ i \in \mathbf{n-1}$.

Note that all derivatives in these n−1 equations are evaluated at the Nash solution. To simplify notation we will drop this argument whenever it is the Nash solution. So, unless stated differently, we assume from now on that the argument in the derivatives will always be the Nash solution.

Remark 1 Recall from the two-player case that the Nash solution can geometrically also be characterized as that point x^N on the curve φ such that the line tangent to φ at x^N intersects the d_1-axis at the point $d_1 + 2(x_1^N - d_1)$. For the multi-player case this generalizes as follows. Consider the plane tangent to the graph of φ at x, i.e.,

$$y_n(y_-) = \varphi(x_-) + \varphi'(x_-)(y_- - x_-). \tag{2}$$

Then x^N is the point such that this plane intersects the vertical plane through the Nash point parallel to the d_i-axis in the d_n-plane at the point $y_- = (x_1^N, \ldots, x_{i-1}^N, d_i + 2(x_i^N - d_i), x_{i+1}^N, \ldots, x_{n-1}^N)$, $i \in \mathbf{n-1}$. Substitution of this into (2) yields the equations (1).

Since the solution of the above optimization problem is a maximum location, the second order derivative H of f evaluated at the Nash solution is semi-negative definite. Simple calculations show that

$$H = Dg' \tag{3}$$

where the ith entry, d_{ii}, of the diagonal matrix D is $\Pi_{j\neq i\in\mathbf{n-1}}(x_j^N - d_j)$ and

$$g'_{x_-}\left(x_-^N, d\right) = \frac{\partial g}{\partial x_-}\left(x_-^N, d\right) = \left(ee^T + I\right)\operatorname{diag}\left(\varphi'_i\right) + \operatorname{diag}\left(x_i^N - d_i\right)\varphi''. \tag{4}$$

We assume throughout this note additionally that H is invertible. In particular, it follows then from (3) that the inverse of g' exists and $g'^{-1} = H^{-1}D$. According to the implicit function theorem

$$\frac{\partial x_-^N}{\partial d} = -\left\{\frac{\partial g}{\partial x_-}(x_-^N, d)\right\}^{-1} \frac{\partial g}{\partial d}.$$

It is easily verified that

$$\frac{\partial g}{\partial d} = -(\mathrm{diag}(\varphi_i^{'}) \mid e). \tag{5}$$

To complete the picture of $\frac{\partial x_i^N}{\partial d_j}$, we still have to consider $\frac{\partial x_n^N}{\partial d_j}$. To that end, we recall that $x_n^N = \varphi(x_-^N)$. Consequently,

$$\frac{\partial x_n^N}{\partial d_j} = \varphi^{'} \begin{pmatrix} \frac{\partial x_1^N}{\partial d_j} \\ \vdots \\ \frac{\partial x_{n-1}^N}{\partial d_j} \end{pmatrix}, \text{ where } \varphi' := \left(\varphi'_1, \ldots, \varphi'_{n-1}\right).$$

So, with $L := \begin{pmatrix} I \\ \varphi' \end{pmatrix}$, we have that $\frac{\partial x^N}{\partial d} = -L\left\{\frac{\partial g}{\partial x_-}(x_-^N, d)\right\}^{-1} \frac{\partial g}{\partial d}$.

Before we present the sensitivity matrix we introduce for notational convenience

$$v_i^N := \frac{x_i^N - d_i}{\sqrt{\varphi(x_-^N) - d_n}} \text{ and } G := \left(-(ee^T + I) + (\varphi - d_n)\mathrm{diag}\left(\frac{1}{\varphi_i^{'}}\right)\varphi^{''}\mathrm{diag}\left(\frac{1}{\varphi_i^{'}}\right)\right)^{-1}.$$

Theorem 1 *If the hamiltonian of the Pareto frontier evaluated at the Nash solution is invertible, the sensitivity matrix of the Nash solution is given by*

$$\frac{\partial x^N}{\partial d} = -\begin{pmatrix} \mathrm{diag}(v_i^N) \\ -v_n^N e^T \end{pmatrix} G \left(\mathrm{diag}\left(\frac{1}{v_i^N}\right) \mid \frac{-1}{v_n^N} e\right). \tag{6}$$

Proof Using (4,5) it is clear that

$$\frac{\partial x^N}{\partial d} = \begin{pmatrix} I \\ \varphi' \end{pmatrix} \left((ee^T + I)\mathrm{diag}(\varphi_i^{'}) + \mathrm{diag}(x_i^N - d_i)\varphi^{''}\right)^{-1} (\mathrm{diag}(\varphi_i^{'}) \mid e). \tag{7}$$

Elementary rewriting of this Eq. (7) gives:

$$\begin{aligned}\frac{\partial x^N}{\partial d} = \begin{pmatrix} I \\ \varphi' \end{pmatrix} &\left((ee^T + I)\mathrm{diag}\left(\frac{\varphi_i^{'}(x_i^N - d_i)}{x_i^N - d_i} \right) \right. \\ &\left. + \mathrm{diag}(x_i^N - d_i)\varphi^{''}\mathrm{diag}(x_i^N - d_i)\mathrm{diag}\left(\frac{1}{x_i^N - d_i} \right) \right)^{-1} \\ &\times \left(\mathrm{diag}\left(\frac{\varphi_i^{'}(x_i^N - d_i)}{x_i^N - d_i} \right) \mid e \right).\end{aligned}$$

From (1) we have that at the Nash solution

$$\varphi_i^{'}(x_i^N - d_i) = \varphi_j^{'}(x_j^N - d_j) = -(\varphi - d_n). \tag{8}$$

Using this, we can rewrite the above equation as follows

$$\begin{aligned}\frac{\partial x^N}{\partial d} &= \begin{pmatrix} I \\ \varphi' \end{pmatrix} \mathrm{diag}(x_i^N - d_i) \left(-(\varphi - d_n)(ee^T + I) + \mathrm{diag}(x_i^N - d_i)\varphi^{''}\mathrm{diag}(x_i^N - d_i) \right)^{-1} \\ &\quad \times \left(-(\varphi - d_n)\mathrm{diag}\left(\frac{1}{x_i^N - d_i} \right) \mid e \right) \\ &= \begin{pmatrix} \mathrm{diag}(v_i^N) \\ -v_n^N e^T \end{pmatrix} \left(-(ee^T + I) + \mathrm{diag}(v_i^N)\varphi^{''}\mathrm{diag}(v_i^N) \right)^{-1} \left(-\mathrm{diag}\left(\frac{1}{v_i^N} \right) \mid \frac{1}{v_n^N} e \right) \\ &= \begin{pmatrix} \mathrm{diag}(v_i^N) \\ -v_n^N e^T \end{pmatrix} \left(-(ee^T + I) + \mathrm{diag}\left(\frac{\varphi_i^{'} v_i^N}{\varphi_i^{'}} \right) \varphi^{''} \mathrm{diag}\left(\frac{\varphi_i^{'} v_i^N}{\varphi_i^{'}} \right) \right)^{-1} \\ &\quad \times \left(-\mathrm{diag}\left(\frac{1}{v_i^N} \right) \mid \frac{1}{v_n^N} e \right).\end{aligned}$$

From this, using (8) and the above introduced notation, (6) is obtained. □

Elementary rewriting of (6) shows that the sensitivity matrix can also be written as

$$\frac{\partial x^N}{\partial d} = - \begin{pmatrix} \mathrm{diag}(v_i^N) G \mathrm{diag}\left(\frac{1}{v_i^N} \right) & \frac{-1}{v_n^N}\mathrm{diag}(v_i^N) G e \\ -v_n^N e^T G \mathrm{diag}\left(\frac{1}{v_i^N} \right) & e^T G e \end{pmatrix}. \tag{9}$$

Since, by assumption, $\varphi^{''}$ is negative definite, G is negative definite too. Thus it follows immediately from (9) that all diagonal entries of the sensitivity matrix are positive. Or stated differently,

Corollary 1 *The Nash solution is d-monotonic.*

Next, we address the question under which conditions on φ the Nash solution is locally strongly d-monotonic. We have the following result:

Theorem 2 *The Nash solution is locally strongly d-monotonic if and only if $-G$ is a diagonally dominant Stieltjes matrix.*

Proof Consider (9). Since $v_i^N > 0$, it follows that $\operatorname{sgn}((\frac{\partial x^N}{\partial d})_{ij}) = \operatorname{sgn}(-G_{ij})$, $i, j \in \mathbf{n-1}$. As already noted, $-G$ is positive definite. So, by Lemma 1.2, $-G$ is a Stieltjes matrix. Moreover it follows from (9) that $\operatorname{sgn}((\frac{\partial x^N}{\partial d})_{in}) = \operatorname{sgn}(\frac{v_i^N}{v_n^N} e_i^T Ge)$. So, $(\frac{\partial x^N}{\partial d})_{in} \leq 0$ if and only if entry i of $Ge \leq 0$, for all $i \in \mathbf{n-1}$. Or, stated differently, $-G$ is diagonally dominant. □

Remark 2 (1) Since $G = -\operatorname{diag}(\varphi_i^{'}) \left(g_{x_-}^{'}(x_-^N, d)\right)^{-1}$, $-G^{-1}$ is nonnegative if and only if $g_{x_-}^{'}(x_-^N, d)$ is nonpositive. Furthermore, as already noticed, $-G$ is positive definite. Therefore, an equivalent statement of Theorem 2 is: the Nash solution is locally strongly d-monotonic if and only if $g_{x_-}^{'}(x_-^N, d)$ is nonpositive and $\operatorname{diag}(\varphi_i^{'}) \left(g_{x_-}^{'}(x_-^N, d)\right)^{-1}$ is a diagonally dominant matrix with nonpositive off-diagonal entries.
This clarifies the statement about ultrametric matrices we made at the end of the preliminaries.
Notice that $g_{x_-}^{'}(x_-^N, d)$ is nonpositive if, e.g., $\varphi^{''}$ is a nonpositive matrix.

(2) Since $-G$ is an M-matrix it follows that $ee^T - (\varphi - d_n)\operatorname{diag}\left(\frac{1}{\varphi_i^{'}}\right) \varphi^{''} \operatorname{diag}\left(\frac{1}{\varphi_i^{'}}\right)$ is nonnegative. Or, equivalently, $\varphi'^T \varphi' - (\varphi - d_n)\varphi^{''}$ is a nonnegative matrix.

(3) In the two-player case, the Nash solution is always strongly d-monotonic.
In the three-player case the Nash solution is locally strongly d-monotonic if and only if $g_{x_-}^{'}(x_-^N, d)$ is nonpositive and $g_{x_-}^{'}(x_-^N, d)\operatorname{diag}\left(\frac{1}{\varphi_i^{'}}\right)$ is a diagonally dominant matrix. This follows directly from Remark 2.1 by a simple spelling of $-G$.

(4) In case the Pareto frontier has no "extreme bendings", i.e. $\varphi^{'}$ is "almost" constant, $\varphi^{''}$ is "almost" zero. In that case $-G$ approximately equals $(ee^T + I)^{-1}$ which is a diagonally dominant Stieltjes matrix. Since this is the case independent of the choice of the threat-point, under these conditions the Nash solution will be strongly d-monotonic.

Next we derive a sufficient condition on the Pareto frontier under which the Nash solution is (globally) strongly d-monotonic. Inspired by items (1) and (2) of the above remark we consider the case when $\varphi^{''}$ is a nonpositive matrix. The result is stated in Theorem 3. Its proof uses the next lemma.

Lemma 2 *Assume S is an invertible matrix and D is a positive diagonal matrix. Consider $P := (S + D)^{-1}$.*

(1) *If S^{-1} is diagonally dominant, then P is diagonally dominant.*
(2) *If S^{-1} is a Stieltjes matrix, then P is a Stieltjes matrix.*

Proof (1) First notice that

$$(S + D)^{-1} = D^{-1} - D^{-1}(D^{-1} + S^{-1})^{-1}D^{-1}. \tag{10}$$

Next consider

$$H := \begin{pmatrix} D^{-1} + S^{-1} & D^{-1} \\ D^{-1} & D^{-1} \end{pmatrix}.$$

Due to our assumptions, it is easily verified that H is diagonally dominant. From e.g., Lei et al. (2003) (see also Carlson and Markham 1979) we conclude then that the Schur complement of H, which equals (10), is also diagonally dominant.

(2) Since by assumption, S^{-1} is a Stieltjes matrix, by Lemma 1.1, S^{-1} is a positive definite matrix. From this it is obvious that P is positive definite too. So, the diagonal entries of P are positive.
Furthermore since, by assumption, S and D are both nonnegative matrices, so is $S + D$. Next we consider the off-diagonal entries of P. Since both D^{-1} and S^{-1} are Stieltjes matrices, so is $D^{-1} + S^{-1}$. So, in particular, all entries of $(D^{-1} + S^{-1})^{-1}$ are nonnegative. From (10) it is obvious then that all off-diagonal entries of P are nonpositive. Since we already argued above that P is positive definite, Lemma 1.2 shows that P is a Stieltjes matrix. □

Theorem 3 *Assume that at any point of the Pareto frontier*

$$\Phi^{-1} := -\left[\operatorname{diag}\left(\frac{1}{\varphi_i'}\right)\varphi''\operatorname{diag}\left(\frac{1}{\varphi_i'}\right)\right]^{-1} \tag{11}$$

is a diagonally dominant Stieltjes matrix. Then, the Nash solution is strongly d-monotonic.

Proof What has to be shown is that irrespective of the choice of the threat-point d the Nash solution is locally strongly d-monotonic or, equivalently (see Theorem 2), that $-G$ is a diagonally dominant Stieltjes matrix.

To that end first note that, since Φ^{-1} is a diagonally dominant Stieltjes matrix, so is $(\varphi - d_n)\Phi^{-1}$. Therefore, by Lemma 2, so is

$$P := (I + (\varphi - d_n)\Phi)^{-1}. \tag{12}$$

Furthermore,

$$-G = \left((ee^T + I) + (\varphi - d_n)\Phi\right)^{-1} = \left(ee^T + P^{-1}\right)^{-1} = P - Pe(e^TPe + 1)^{-1}e^TP.$$

Since P is diagonally dominant, $Pe \geq 0$. Consequently, $Pe(e^TPe + 1)^{-1}e^TP \geq 0$. So, all off-diagonal entries of $-G$ are nonpositive. Obviously, $-G$ is positive definite and all entries of $-G^{-1}$ are nonnegative. So, by Lemma 1.2, $-G$ is a Stieltjes matrix. Furthermore it follows from (13) that

$$-Ge = \left(P - Pe(e^T Pe + 1)^{-1} e^T P\right) e = \left(1 - \frac{e^T Pe}{1 + e^T Pe}\right) Pe \geq 0.$$

That is, $-G$ is diagonally dominant. □

Remark 3 (1) Clearly, (11) is only satisfied if $\varphi^{''}$ is nonpositive. Furthermore, it is easily verified that for the scalar case $\Phi = [\frac{1}{\varphi^{'}}]^{'}$, whereas for the multivariable case, with $S := \text{diag}(\varphi_i^{'})$, $\Phi = S \left[\frac{1}{\varphi_i^{'}} \cdots \frac{1}{\varphi_{n-1}^{'}} \right]^{'} S^{-1}$.

This relationship might be helpful in getting a better intuition about the conditions under which a bargaining solution satisfies strong d-monotonicity.

(2) Consider the next statements:
   (i) Φ^{-1} is a diagonally dominant Stieltjes matrix.
   (ii) $(ee^T + (\varphi - d_n)\Phi)^{-1}$ is a diagonally dominant Stieltjes matrix.
   (iii) $-G$ is a diagonally dominant Stieltjes matrix.

   Then, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). The first implication can be shown by using the fact that $(ee^T + \Phi)^{-1} = \frac{1}{\alpha}\Phi^{-1}(\alpha I - ee^T \Phi^{-1})$, where $\alpha = 1 + \sum_{i,j=1}^{n} \Phi_{i,j}^{-1} > 0$, whereas the second implication follows using similar arguments as in the proof of Theorem 3. Unfortunately none of the reverse implications holds.

(3) In particular it follows from (11), using (1), that the Nash solution is locally strongly d-monotonic if $-[\text{diag}(x_i^N - d_i)\varphi^{''}(x_-^N)\text{diag}(x_i^N - d_i)]^{-1}$ is a diagonally dominant Stieltjes matrix.

Remark 4 The above analysis can also be used to study the *weighted Nash solutions*. That is the argument that solves the maximization problem

$$\max_{x_-} \tilde{f}(x_-) := \max_{x_-} \Pi_{i \in \mathbf{n-1}} (x_i - d_i)^{\alpha_i} (\varphi(x_-) - d_n)^{\alpha_n},$$

where the positive weights α_i reflect the agent's bargaining powers.

Introducing the weight matrix $W_k \in \mathbb{R}^{k \times k}$ as $W_k := \text{diag}(\frac{\alpha_n}{\alpha_i})$ we have that with

$$\tilde{G} := \left(-(ee^T + W_{n-1}) + (\varphi - d_n)\text{diag}\left(\frac{1}{\varphi_i^{'}}\right)\varphi^{''}\text{diag}\left(\frac{1}{\varphi_i^{'}}\right)\right)^{-1},$$

$$\frac{\partial x^N}{\partial d} = -W_n \begin{pmatrix} \text{diag}(v_i^N) \\ -v_n^N e^T \end{pmatrix} \tilde{G} \left(\text{diag}\left(\frac{1}{v_i^N}\right) \mid \frac{-1}{v_n^N} e \right).$$

Using this it follows then that the results of Corollary 1 and Theorems 2 and 3 apply for this case as well, with G replaced by $\tilde{G}$.

4 Examples

In this section we provide two examples. The first example provides a number of Pareto frontiers for which the Nash solution is strongly d-monotonic. Intuitively it demonstrates that if the frontier "does not bend too much" one may expect that this property holds.

The second example may be interpreted as a cartel-formation game. For different sets of parameters we present numerically the set of threat-points where the Nash solution is locally strongly d-monotonic.

Example 1 (1) Assume that φ'' is a nonpositive diagonal matrix (so, $\varphi(x_-)$ is e.g., a plane or $\varphi(x_-) = r + b^T x_- + \frac{1}{2}x_-^T A x_-$, where b, x_- are $(n-1)$-dimensional vectors with $b \leq 0$ and $d \geq 0$ and A a nonpositive diagonal matrix). Then, for every choice of d, $-G$ is a diagonally dominant Stieltjes matrix. So, see Theorem 2, the Nash solution is strongly d-monotonic.

(2) Assume that the Pareto frontier has a constant curvature, that is

$$\varphi(x_-) = \sqrt{r^2 - x_1^2 - \cdots - x_{n-1}^2}.$$

Then $\varphi_i' = \frac{-x_i}{\varphi(x_-)}$, $\varphi_{ij}'' = \frac{-x_i x_j}{\varphi^3(x_-)}$ if $i \neq j$ and $\varphi_{ii}'' = \frac{-(\varphi^2(x_-)+x_i^2)}{\varphi^3(x_-)}$. Consequently,

$$\Phi = \frac{1}{\varphi(x_-)} \begin{pmatrix} \frac{\varphi^2(x_-)+x_1^2}{x_1^2} & 1 & \cdots & \cdots & 1 \\ 1 & \frac{\varphi^2(x_-)+x_2^2}{x_2^2} & 1 & \cdots & 1 \\ \vdots & \ddots & & & \vdots \\ \vdots & & & \ddots & 1 \\ 1 & \cdots & \cdots & 1 & \frac{\varphi^2(x_-)+x_{n-1}^2}{x_{n-1}^2} \end{pmatrix}$$

and

$$\Phi^{-1} = \frac{1}{r^2\varphi(x_-)} \begin{pmatrix} (r^2 - x_1^2)x_1^2 & -x_1^2 x_2^2 & \cdots & \cdots & -x_1^2 x_{n-1}^2 \\ -x_2^2 x_1^2 & (r^2 - x_2^2)x_2^2 & -x_2^2 x_3^2 & \cdots & x_2^2 x_{n-1}^2 \\ \vdots & \ddots & & & \vdots \\ \vdots & & & \ddots & -x_{n-2}^2 x_{n-1}^2 \\ -x_{n-1}^2 x_1^2 & \cdots & \cdots & -x_{n-1}^2 x_{n-2}^2 & (r^2 - x_{n-1}^2)x_{n-1}^2 \end{pmatrix}.$$

Obviously Φ^{-1} is a Stieltjes matrix. Furthermore $\Phi^{-1}e = \frac{\varphi}{r^2}[x_1^2, \ldots, x_{n-1}^2]^T$. So Φ^{-1} is diagonally dominant too. Therefore (see Theorem 3) the Nash solution is strongly d-monotonic. It is easily verified that this result also holds if $\varphi(x_-)$ is replaced by $\varphi(x_-) = \sqrt{r^2 - \alpha_1 x_1^2 - \cdots - \alpha_{n-1}x_{n-1}^2}$, $\alpha_i > 0$.

(3) Assume that the Pareto frontier is described by $\varphi(x_-) = \Pi_{i=1}^{n-1}(b_i - x_i)^{\alpha_i}$, where $0 < \alpha_i < 1$ and $x_i \leq b_i,\ i \in \mathbf{n-1}$. Note that this type of functions includes e.g., the Cobb-Douglas function which often occurs in economics. Then,

$$\varphi_i^{'} = \frac{-\alpha_i}{b_i - x_i}\varphi;\ \varphi_{ij}^{''} = \frac{\alpha_i\alpha_j}{(b_i - x_i)(b_j - x_j)}\varphi,\ i \neq j;\ \text{and}\ \varphi_{ii}^{''} = \frac{\alpha_i(-1+\alpha_i)}{(b_i - x_i)^2}\varphi.$$

Elementary calculations show that then $-\Phi = \frac{1}{\varphi}\left(ee^T + \operatorname{diag}\left(\frac{-1}{\alpha_i}\right)\right)$. Consequently,

$$\begin{aligned}
-G &= \left(ee^T + I + (\varphi - d_n)\Phi\right)^{-1} \\
&= \left(ee^T + I - (\varphi - d_n)\frac{1}{\varphi}\left(ee^T + \operatorname{diag}\left(\frac{-1}{\alpha_i}\right)\right)\right)^{-1} \\
&= \varphi\left(d_n ee^T + \operatorname{diag}\left(\varphi + \frac{\varphi - d_n}{\alpha_i}\right)\right)^{-1} \\
&= \varphi\left(D^{-1} - D^{-1}e(e^T D^{-1}e + \frac{1}{d_n})^{-1}e^T D^{-1}\right),
\end{aligned} \tag{13}$$

where $D := \operatorname{diag}(\varphi + \frac{\varphi - d_n}{\alpha_i})$. From (13) it is easily verified that $-G$ is a Stieltjes matrix. Furthermore $-Ge = (1 - \frac{e^T D^{-1}e}{e^T D^{-1}e + \frac{1}{d_n}})D^{-1}e$. Clearly this vector is positive, so $-G$ is diagonally dominant too. Since, for all d, $-G$ is a diagonally dominant Stieltjes matrix, we conclude that the Nash solution is strongly d-monotonic.

Example 2 Consider three firms who sell an amount x_i of a product on a market. The price, p, they get on the market depends on the quantity sold by all firms. That is, $p = c - \alpha_1 x_1 - \alpha_2 x_2 - \alpha_3 x_3$, where α_i and c are some positive constants. The costs for producing x_i are $C_i(x_i)$. So the profits for firm i are $\pi_i = px_i - C_i(x_i)$. Next consider the parameterized joint profit function

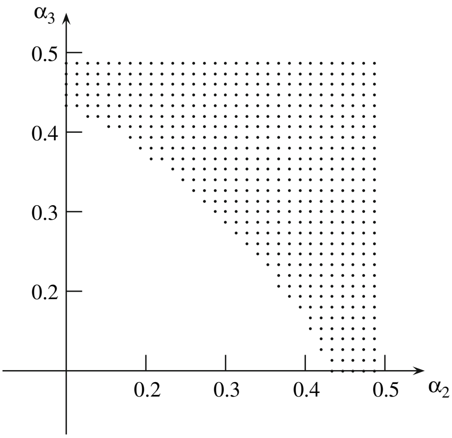
$$\pi = \sum_{i=1}^{3} \lambda_i \pi_i, \quad \text{where} \sum_{i=1}^{3} \lambda_i = 1 \quad \text{and} \quad \lambda_i \geq 0.$$

By maximizing π for all possible parameter combinations, $\lambda := (\lambda_1, \lambda_2, \lambda_3)$, one obtains the Pareto frontier characterizing all possible joint maximal profits the firms can obtain by cooperation.

For simplicity we will assume in this example that the threat-point is given exogenously. This assumption might make sense, e.g., in a situation where the firms are divisions of a parent company.

We will consider two different specifications for the cost functions C_i.

Fig. 1 Case $\alpha_1 = 1$, $\beta_i = 1$, $c = 5$, $d = 0$. Dot=locally strongly d-monotonic



Case 1 $C_i(x_i) = \beta x_i$.

In this case straightforward (though lengthy) calculations show that the Pareto frontier is given by the plane:

$$\alpha_1\pi_i + \alpha_2\pi_2 + \alpha_3\pi_3 = \left(\frac{c-\beta}{2}\right)^2.$$

So, for all d, the Nash solution is strongly d-monotonic in this case (see Example 1.1).

Case 2 $C_i(x_i) = \beta_i x_i^2$.

In this case it is not possible to derive an analytic expression for the Pareto frontier. We will briefly indicate how one may pursue this case numerically to verify the local strong d-monotonicity of the Nash solution. Differentiation of π w.r.t. x_i yields 3 first order conditions in x_i for every λ (in this case, this is just a set of linear equations). From this, one can solve x_i in terms of λ. Then, one can determine for an arbitrary threat-point the corresponding Nash solution, with e.g., the numerical algorithm outlined in Douven (1995, Sect. 3.3.2), (see also Engwerda 2005, Sect. 6.4).

From the seven equations $\pi_i - px_i - C_i(x_i) = 0$, $\frac{\partial\pi}{\partial x_i} = 0$, $i = 1, 2, 3$ and $\sum_{i=1}^{3}\lambda_i = 1$, one can then implicitly solve $(\pi_3, x_1, x_2, x_3, \lambda_1, \lambda_2, \lambda_3)$ as a function of π_1 and π_2. In particular the implicit function theorem can be used to find analytic expressions for the derivative and hamiltonian of $\pi_3 = \varphi(\pi_1, \pi_2)$ at the Nash solution. From this, the monotonicity property can straightforwardly be verified.

Figures 1, 2, 3 present some results for this example for different parameters and threat-points. In all these cases, the parameters $\alpha_1 = 1$, $\beta_i = 1$ and $c = 5$ remained unchanged.

Figure 1 reports the local strong d-monotonicity property of the Nash solution, if the threat-point $d = 0$, for different values of the parameters α_2 and α_3 (both ranging between 0.1 and 0.5). A dot (empty space) indicates that with that choice of parameters, the zero-threat-point is (not) locally strongly d-monotonic. Not shown here is

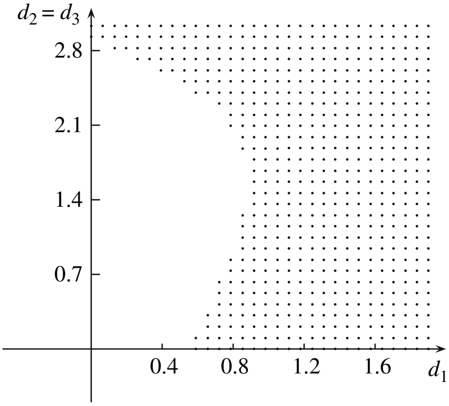


Fig. 2 Case $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = \frac{1}{4}$, $\beta_i = 1$, $c = 5$

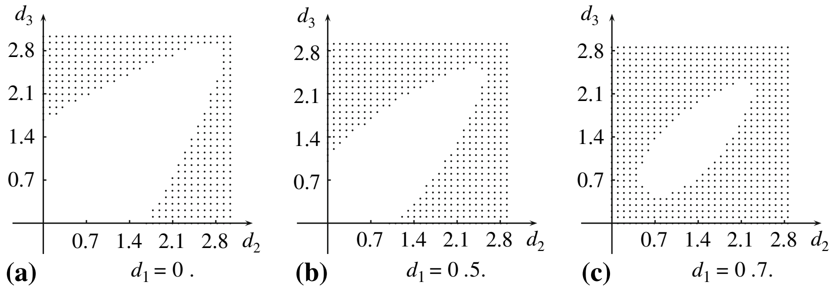


Fig. 3 Case $\alpha_1 = 1, \alpha_2 = \alpha_3 = \frac{1}{4}, \beta_i = 1, c = 5$

that for large values of these parameters the zero-threat-point is also locally strongly d-monotonic. So the case of two firms having a small impact on the price compared to the third firm seems to not be stable (in the sense discussed in the introduction). Here the notion "small" should however be interpreted in the light of the other model parameters that were kept constant. Some additional experiments suggest that the level of the β_i parameters are more important than that of the c parameter for this comparison. If all firms have a substantial effect on the price, the Nash solution is locally strongly d-monotonic.

Figure 2 reports local strong d-monotonicity for different threat-points in case $\alpha_2 = \alpha_3 = \frac{1}{4}$. The threat-point d_1 ranges here from 0 to 1.9 and $d_2 = d_3$ ranges from 0 to 3.1. To complete the three dimensional picture we plotted in Fig. 3 for three different values of d_1 (0, 0.5 and 0.7, respectively) using the same model parameters the monotonicity result if the other two coordinates of the threat-point d_2 and d_3 range between 0 and 3.1. Notice that the closer the threat-point is to the Pareto frontier, the more this frontier resembles a plane. For that reason in fact the dots in the right and upper part of these graphs extend until the Pareto frontier. For numerical simplicity we did not plot this extension.

Concluding, this example demonstrates that if firms have a substantial effect on the price, the Nash solution is strong d-monotonic. If at least one firm has a "small" (see above discussion) impact on the price, there exist areas of threat-points where the

Nash solution is not locally strongly d-monotonic. Furthermore we observe that if at a certain threat-point the Nash solution is locally strongly d-monotonic this does not imply that at every larger threat-point the Nash solution has this property too.

5 Concluding remarks

In this note we derived, under some technical conditions, the sensitivity matrix of the Nash bargaining solution w.r.t. the disagreement point d. In particular, this makes it possible to analyze the local strong d-monotonicity of the Nash solution. We showed that the Nash solution satisfies this property if and only if a certain matrix, $-G$, evaluated at the Nash bargaining solution is a diagonally dominant Stieltjes matrix. Using this result, a class of bargaining problems was characterized for which the Nash solution is strongly d-monotonic. The results were illustrated in a number of examples.

The condition under which the Nash solution is locally strongly d-monotonic is phrased in terms of the (second) order derivative of the Pareto frontier. Unfortunately at this moment a clear intuition about the set of problems for which the Nash solution is (locally) strongly d-monotonic is lacking. From the condition it is clear that if the Pareto frontier has no extreme bendings, the property holds. This implies that if in the bargaining problem, the interests of the players are similar the Nash solution is locally strongly d-monotonic. Finding a geometric interpretation of the conditions and, from that, more intuition about the set of bargaining problems for which the Nash solution satisfies the monotonicity property remains, however, an open problem.

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